Lecture 17

8.3 - Moments and Centers of Mass

1 dimension (discrete case)

(This case does not even require calculus.)

Suppose we have a system of point masses $m_1, m_2, \ldots, m_n$ which are placed on the x-axis at points $x_1, x_2, \ldots, x_n$. Then, the moment of the system about the origin is

$$\boxed{M_0 = m_1x_1 + m_2x_2 + \cdots + m_nx_n}$$

and the center of mass of the system is

$$\boxed{\bar{x} = \frac{M_0}{m}, \qquad m = m_1 + \cdots + m_n}$$

If you imagine the x-axis above as a rod of negligible mass, then the center of mass is the point on the rod where you could balance the rod on your fingertip. The moment measures the rods tendency to rotate if you try to balance it at the origin: clockwise if $M_0 > 0$, counterclockwise if $M_0 < 0$, balanced if $M_0 = 0$.

If we replace the system with one where all of the mass is concentrated at the center of mass, then it will have the same moment as the original system.

2 dimensions (discrete case)

This time, our masses $m_1, \dots, m_n$ are distributed around the xy-plane at points $(x_1, y_1), \dots, (x_n, y_n)$. This time, we have moments about the x- & y- axes:

Moment about y-axis: $M_y = m_1 x_1 + \cdots + m_n x_n$

Moment about x-axis: $M_x = m_1 y_1 + \cdots + m_n y_n$

The center of mass of this system is the point $(\bar{x}, \bar{y})$ in the plane given by:

$$\boxed{\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}, \quad m = m_1 + \cdots + m_n}$$

Of course, not many systems are discrete, Calculus allows us to deal with this case:

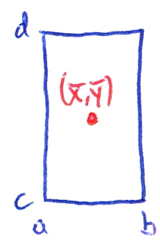
2 dimensions (continuous)

Suppose we have a region R in the plane with uniform (constant) density (we need double integrals to deal with non-uniform density, c.f., calc III); we can find the mass, moments about the x- & y-axes, and center of mass of this object too.

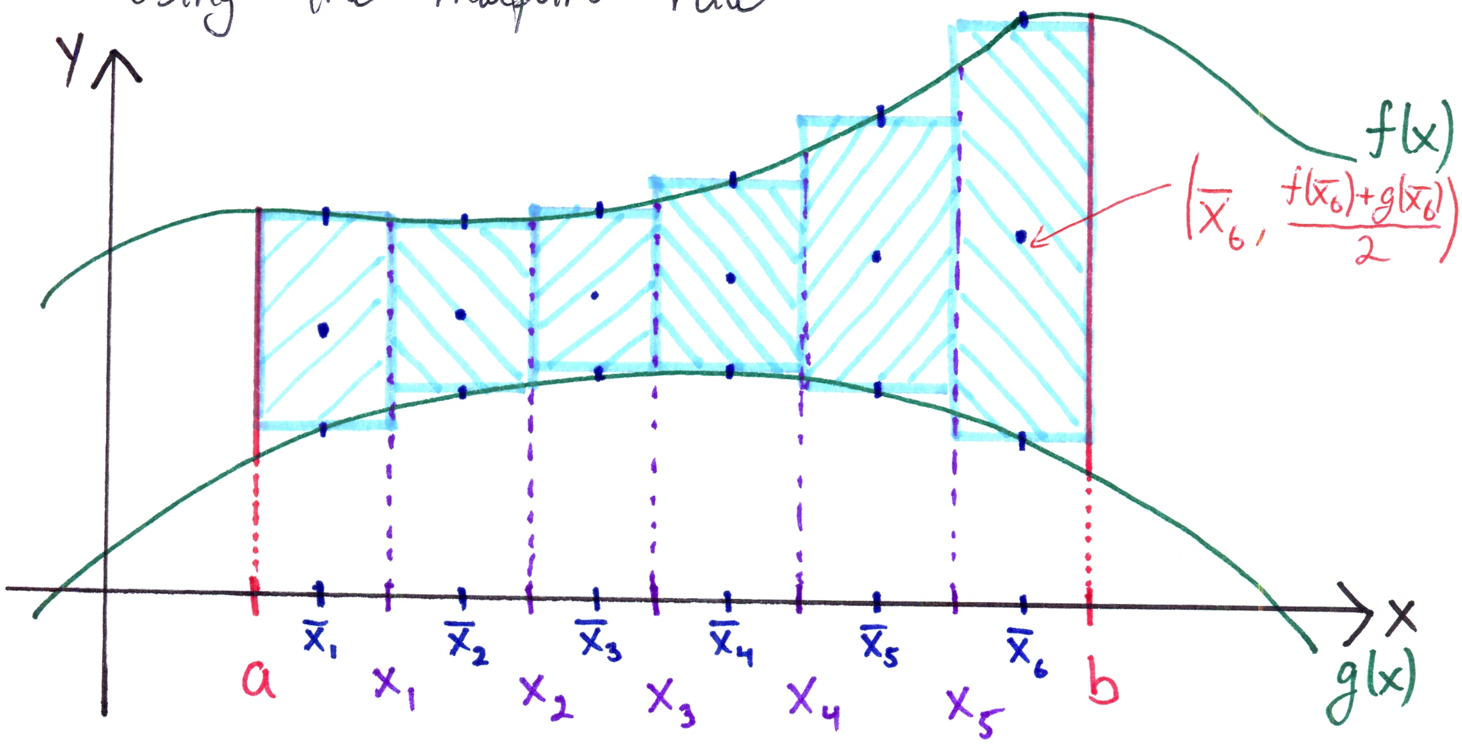
Let's suppose R is a region between between two functions $y=f(x)$ & $y=g(x)$, i.e.,

$$R=\{(x,y)\mid a\le x\le b,\ g(x)\le y\le f(x)\},$$

and that R has constant density ρ. We compute the mass and moments using Riemann sums to find the integrals. Riemann sums use rectangles, so first we need to know where the center of mass of a rectangle with constant density is. It isn't too hard to believe that the center of mass of a rectangle $[a,b]\times[c,d]$ is its "center" $\left(\frac{a+b}{2},\frac{c+d}{2}\right)$.



Since the center of mass is in the middle of the rectangle, we should set up the Riemann sums using the midpoint rule:



First, the mass: (mass = density * area)

$$m = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \cdot \left(\begin{smallmatrix}\text{area of} \\ i^{th}\text{ rectangle}\end{smallmatrix}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{\rho\left(f(\bar{x}_i) - g(\bar{x}_i)\right)\Delta x}_{\substack{= \\ m_i \\ \text{(mass of } i^{th} \text{ rectangle)}}} = \int_a^b \rho\left(f(x) - g(x)\right) dx$$

The moments about the x- & y- axes:

$$M_y = \lim_{n\to\infty} \sum_{i=1}^{n} m_i \bar{x}_i = \lim_{n\to\infty} \sum_{i=1}^{n} \rho \bar{x}_i \left(f(\bar{x}_i) - g(\bar{x}_i)\right)\Delta x = \int_a^b \rho x \left(f(x) - g(x)\right) dx$$

$$M_x = \lim_{n\to\infty} \sum_{i=1}^{n} m_i \left(\frac{f(\bar{x}_i) + g(\bar{x}_i)}{2}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\rho}{2} \left(f(\bar{x}_i) - g(\bar{x}_i)\right)\left(f(\bar{x}_i) + g(\bar{x}_i)\right)\Delta x$$

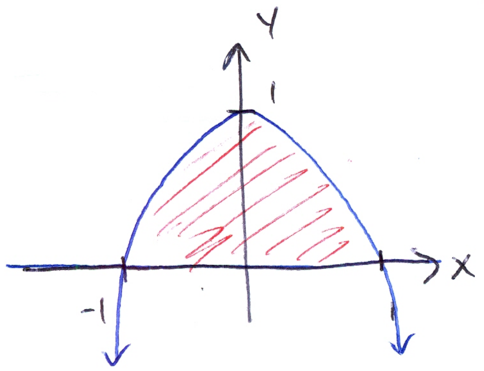
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\rho}{2}\left[(f(\bar{x}_i))^2 - (g(\bar{x}_i))^2\right]\Delta x = \int_a^b \frac{\rho}{2}\left[(f(x))^2 - (g(x))^2\right] dx$$

Of course, the center of mass, as before, is given by $(\bar{x},\bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$

Notice that the ρ from each integral cancels, i.e., $(\bar{x},\bar{y})$ is independent of ρ.

<u>Ex</u>: Find the centroid (another name for center of mass) of the region bounded by $y=1-x^2$ and the x-axis.

<u>Sol</u>: The x-axis is given by $y=0$.



$$\text{mass} = m = \rho\int_{-1}^{1}(1-x^2)\,dx = \rho\left(x-\tfrac{1}{3}x^3\right)\Big|_{-1}^{1}$$

$$= \rho\left[\tfrac{2}{3}-\left(-\tfrac{2}{3}\right)\right] = \frac{4\rho}{3}$$

<u>Moment about y-axis</u>:

$$M_y = \rho\int_{-1}^{1} x(1-x^2)\,dx = \rho\int_{-1}^{1}(x-x^3)\,dx = \rho\left(\frac{x^2}{2}-\frac{x^4}{4}\right)\Big|_{-1}^{1} = 0$$

<u>Moment about x-axis</u>:

$$M_x = \frac{\rho}{2}\int_{-1}^{1}(1-x^2)^2\,dx = \frac{\rho}{2}\int_{-1}^{1}(1-2x^2+x^4)\,dx = \frac{\rho}{2}\left(x-\tfrac{2}{3}x^3+\tfrac{1}{5}x^5\right)\Big|_{-1}^{1}$$

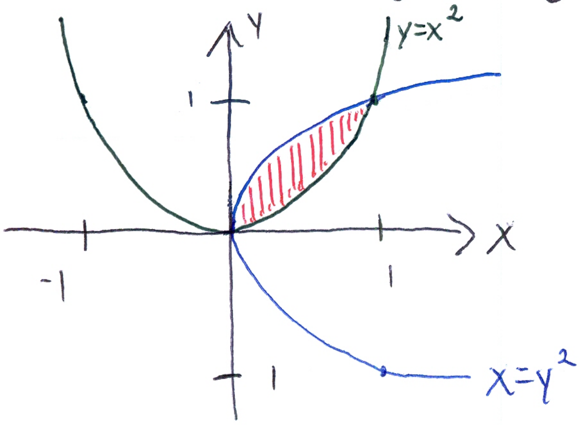
$$= \frac{\rho}{2}\left[\left(1-\tfrac{2}{3}+\tfrac{1}{5}\right)-\left(-1+\tfrac{2}{3}-\tfrac{1}{5}\right)\right] = \frac{8\rho}{15}$$

<u>Centroid</u>: $$(\bar{x},\bar{y}) = \left(\frac{M_y}{m},\frac{M_x}{m}\right) = \left(0, \frac{\frac{8\rho}{15}}{\frac{4\rho}{3}}\right) = \left(0,\frac{2}{5}\right)$$

We can also see this by symmetry since R is symmetric about the y-axis.

Ex: Find the mass and center of mass of the region between the curves $y=x^2$ and $x=y^2$ if it has constant density $\rho=6$.

Sol:



The region sits above $y=x^2$ and below $y=\sqrt{x}$.

$$\text{mass} = 6\int_0^1 (\sqrt{x}-x^2)\,dx = 6\left(\tfrac{2}{3}x^{3/2}-\tfrac{1}{3}x^3\right)\Big|_0^1 = 6\cdot\tfrac{1}{3} = 2$$

$$M_y = 6\int_0^1 x(\sqrt{x}-x^2)\,dx = 6\int_0^1 (x^{3/2}-x^3)\,dx = 6\left(\tfrac{2}{5}x^{5/2}-\tfrac{1}{4}x^4\right)\Big|_0^1$$

$$= 6\left(\tfrac{2}{5}-\tfrac{1}{4}\right) = 6\left(\tfrac{8-5}{20}\right) = \tfrac{18}{20} = \tfrac{9}{10}$$

$$M_x = \tfrac{6}{2}\int_0^1 \left[(\sqrt{x})^2-(x^2)^2\right]dx = 3\int_0^1 (x-x^4)\,dx = 3\left(\tfrac{1}{2}x^2-\tfrac{1}{5}x^5\right)\Big|_0^1$$

$$= 3\left(\tfrac{1}{2}-\tfrac{1}{5}\right) = 3\left(\tfrac{5-2}{10}\right) = \tfrac{9}{10}$$

$$(\bar{x},\bar{y}) = \left(\frac{9/10}{2}, \frac{9/10}{2}\right) = \left(\frac{9}{20}, \frac{9}{20}\right)$$